New examples of conservative systems on S^2 possessing an integral cubic in momenta

Elena N. Selivanova*

1 Introduction

It has been proved that on 2-dimensional orientable compact manifolds of genus $g > 1$ there is no integrable geodesic flow with an integral polynomial in momenta, see [5, 6]. There is a conjecture that all integrable geodesic flows on T^2 possess an integral quadratic in momenta (for more details see [2]). All geodesic flows on S^2 and T^2 possessing integrals linear and quadratic in momenta have been described in [5, 1], using some ideas going back to Darboux [3]. So far there has been known only one example of conservative system on S^2 possessing an integral cubic in momenta: the case of Goryachev-Chaplygin in the dynamics of a rigid body, see [2].
The aim of this paper is to propose a new one-parameter family of examples of complete integrable conservative systems on S^2 possessing an integral cubic in momenta. We show that our family does not include the case of Goryachev-Chaplygin.
The paper is organized as follows:

We start with the investigation of the following initial value problem

$$u'''u'r^4 = -7u''u'r^3 - 2u''^2r^4 + u''ur^2 - 2u'^2r^2 + u'ur + u^2,$$
$$u(1) = 0, u'(1) = 1, u''(1) = \tau - 1, \tag{1}$$

where $u : \mathbf{R}^+ \mapsto \mathbf{R}$, $r \mapsto u(r)$, and prove that there exists a positive constant T such that the one-parameter family $\{\Psi_\tau\}_{\tau\in(0,T)}$ of solutions of this problem defines a one-parameter family of smooth conservative systems on S^2 with energy

$$H_\tau(r, \varphi, dr, d\varphi) = \frac{r^2 d\varphi^2 + dr^2}{r^4 \Psi_\tau'^2(r)} - \left(\Psi_\tau'' r^2 + \Psi_\tau' r - \Psi_\tau\right) \Psi_\tau'^2 r^2 \cos\varphi \tag{2}$$

in polar coordinates r and φ, possessing nontrivial additional integrals cubic in momenta.
We note that the limit point of our family for $\tau = 0$ corresponds to the function $\Psi_0 = \frac{1}{2}\left(r - \frac{1}{r}\right)$ which defines by (2) the standard metric of constant curvature $K = 1$ on S^2.

*Supported by DAAD.

2 Existence of C^∞ solutions

In this chapter the problem of the existence of the smooth solutions of (1) on $(0,+\infty)$ will be considered.
Initial value problem (1) can be replaced by the initial value problem

$$x'x''' = xx'' - 2x''^2 + x'^2 + x^2,\ x(0) = 0, x'(0) = 1, x''(0) = \tau. \tag{3}$$

This reduction is accomplished by the change of variable $r \to t = \log r$ which changes (1) to (3). Initial value problem (3) is autonomous and more convenient for further investigations. For this reason, the theorem which follows will be stated for (1), but proved for (3).

Theorem 2.1 *There is a positive constant T such that solutions of (1) exist on $(0,+\infty)$ for $\tau \in (-T,T)$.*

Proof. Initial value problem (3) has a unique solution $x(t) = \Theta_\tau(t)$ which is positive on $(0,\varepsilon)$ and negative on $(-\varepsilon,0)$ for a sufficiently small ε.

Let us consider the case $t > 0$.
The differential equation from (3) can be replaced by the following differential equation of the second order

$$\ddot{q} = \frac{1}{q}\left(1 + 2q^2 - 3q^4 + \dot{q} - 7q^2\dot{q} - 2\dot{q}^2\right)$$

with function $q(t) = R'(t)$ where $R(t) = \log x(t)$. We may rewrite this equation as a system of differential equations of the first order:

$$\dot{q} = p,\quad \dot{p} = \frac{1}{q}\left(1 + 2q^2 - 3q^4 + p - 7q^2p - 2p^2\right). \tag{4}$$

Since system (4) is symmetric with respect to $q \mapsto -q$, $t \mapsto -t$, it suffices to consider the case $q > 0$.
In order to obtain the phase portrait of (4) we may consider the following smooth system

$$\dot{q} = qp,\quad \dot{p} = 1 + 2q^2 - 3q^4 + p - 7q^2p - 2p^2. \tag{5}$$

The solutions of (4) are obtained from the solutions of (5) by a reparametrization. System (5) has four singular points: two saddle points $p = 1, q = 0$, $p = -\frac{1}{2}, q = 0$ and two knots $p = 0, q = \pm 1$.

The aim of our further investigations is to show that the orbits of (4), corresponding to the solutions of (3), when τ belongs to a certain interval, converge to the singular point $q = 1, p = 0$.

For any solution $x(t) = \Theta_\tau(t)$, $t \in (0,\varepsilon)$ of (3) there is an orbit $\{\Gamma_\tau : p = p_\tau(q)\}$ in the phase spase of (4).
For any solution of (3) it holds that $R(t) \to -\infty$ as $t \to 0+$ and, therefore, $q(t) =$

$R'(t) \to +\infty$, $p(t) = q'(t) \to -\infty$ as $t \to 0+$. Thus $p_\tau(q) \to -\infty$ as $q \to +\infty$.
We may now consider only the orbits of (4) where $p \to -\infty$ as $q \to +\infty$.
Show that if $\tau_1 > \tau_2$ then $p_{\tau_1}(q) > p_{\tau_2}(q)$. Indeed, for a solution $x(t) = \Theta_{\tau_1}(t)$ of (3) it holds

$$\tau_1 = \lim_{t \to 0+} \frac{\Theta''_{\tau_1}(t)}{\Theta'_{\tau_1}(t)} = \lim_{q \to +\infty} \frac{q^2 + p_{\tau_1}(q)}{q}. \tag{6}$$

Note that the function $x(t) = \sinh t$ satisfies (3) when $\tau = 0$. The related orbit of (4) has then the following form $\{\Gamma_0 : p = p_0(q) = -q^2 + 1\}$.
So, the orbits of (4), corresponding to the solutions of (3), for $\tau \geq 0$ converge to the singular point $q = 1, p = 0$.

We prove below that orbit $(*)$, where $p \to -\frac{1}{2}$ as $q \to 0+$, corresponds to the solution of (3) when τ is equal to a negative constant $-T$ and all orbits of (4), lying between $(*)$ and Γ_0 correspond to the solutions of (3) for $\tau \in (-T, 0)$.

Assume first that there exists a constant $\tau_0 < 0$ such that orbit Γ_{τ_0} does not converge to the point $q = 1, p = 0$.
Consider the set W of the orbits of (4), lying between Γ_{τ_0} and Γ_0. Show that for any orbit in W the value q becomes infinite in a finite time interval.
In fact, for any solution of (4) holds

$$\int^{q(t)} \frac{dq}{p} = t + const. \tag{7}$$

For orbits in W we have $p < -q^2 + 1$. Hence, the left hand side of (7) is bounded for any orbit in W as $q \to +\infty$. Without loss of generality we assume that for these solutions of (4) t vanishes as q becomes infinite.
We conclude that for any orbit from W it holds by (6)

$$(\tau_0 + o(1))q < p + q^2 < 1$$

and for a corresponding solution of (3) we get for $t \to 0+$

$$\tau_0 + o(1) < \frac{x''(t)}{x'(t)} < 1.$$

Thus, for a corresponding solution of (3) the function $(\log x'(t))'$ is bounded in an interval, containing $0+$. Hence, $\lim_{t \to 0+} x'(t)$ is bounded for a solution $x(t)$ of (3), corresponding to an orbit in W and, therefore, $x(t) = \frac{x'(t)}{q} \to 0$ as $t \to 0+$.
Let us consider orbit $(*)$ where $p = p^*(q)$. If $(*)$ is the same as Γ_{τ_0} then $T = -\tau_0$.
If $(*)$ is not the same as Γ_{τ_0} then it belongs to W and, hence, corresponds to the solution of (3) for $\tau = -T$, where T is equal to a positive constant and, moreover, $T < -\tau_0$.
Thus we have shown that for any $\tau \in (-T, T)$ where T is a positive constant sytem (3) has a solution for all $t \geq 0$.

Let us assume now that there is no such orbit Γ_{τ_0} and, so, all orbits of (4), corresponding to the solutions of (3), converge to the singular point $q = 1, p = 0$. Clearly, orbit $(*)$ corresponds to a solution $x(t) = \eta(t)$ of the differential equation in (3). As mentioned above, the function $q(t) = \frac{\eta'(t)}{\eta(t)}$ becomes infinite in a finite time interval of t, say, as $t \to 0+$. Since $p^*(q) < -q^2 - kq$ for any k and for sufficiently large q, for $\eta(t)$ it holds

$$\lim_{t \to 0+} \frac{\eta''(t)}{\eta'(t)} = -\infty. \tag{8}$$

Note that $\eta(0)$ is finite because $R(0) - R(t) = \int_t^0 q(t)dt < 0$ for $t > 0$ and $\eta(t) = \exp R(t)$. So, there are two cases: $\eta(0) = 0$ or $\eta(0) \neq 0$.
Assume that $\eta(0) \neq 0$. It follows that $\eta'(0+) = +\infty$ and from (8) we get $\eta''(0+) = -\infty$.
Rewrite now the differential equation in (3) in the following form

$$\frac{x'''}{x} = -3\frac{x'' - x}{x'} - 2\frac{(x'' - x)^2}{xx'} + \frac{x'}{x}.$$

Therefore, it holds

$$\frac{\eta'''}{\eta} = -3\frac{p^*(q) + q^2 - 1}{q} - 2\frac{(p^*(q) + q^2 - 1)^2}{q} + q.$$

We obtain $\eta'''(0+) = -\infty$. This is a contradiction $\eta''(0+) = -\infty$.
Assume that $\eta(0) = 0$ and $\eta'(0) \neq 0$. From (8) we get $\eta''(0+) = -\infty$. Rewriting the differential equation in (3) in the following form

$$x''' = x' - 3\frac{(x'' - x)x}{x'} - 2\frac{(x'' - x)^2}{x'},$$

we obtain again $\eta'''(0+) = -\infty$.
Therefore, we only have to consider one case $\eta(0) = 0$ and $\eta'(0) = 0$. Taking into account that $\eta'(t) = \eta(t)q(t)$ and $q(t) > 0$ as $t \to 0+$, we conclude that $\eta'(t) \to 0+$ as $t \to 0+$ but on the other hand from (8) it follows that $\eta''(t) < 0$.
These contradictions finally show that there is an orbit Γ_{τ_0} which does not converge to the point $q = 1, p = 0$ and, therefore, for any $\tau \in (-T, T)$, where

$$T = -\lim_{q \to +\infty} \frac{p^*(q) + q^2}{q} < \infty,$$

solutions of (3) exist on $[0, +\infty)$.
Since a solution $x(t) = \Theta_\tau(t)$ of (3) equals $-\Theta_{-\tau}(-t)$ if $t \leq 0$, solutions of (3) exist on $(-\infty, +\infty)$. $\square$

3 Asymptotic behaviour at infinity

In this chapter we deal with the properties of the solutions of (1) for $\tau \in (-T, T)$ at $r = 0$ and $r = +\infty$.

In this chapter initial value problem (1) is oft replaced by the autonomous initial value problem (3) and another initial value problem which is more convenient when $r \to 0$ or $r \to +\infty$.

Theorem 3.1 *There is a constant T such that for any solution $\Psi_\tau(r)$ of (1) with $\tau \in (-T, T)$ it holds*

$$\Psi'_\tau(r) = \frac{1}{r^2}\xi_\tau(r^2) = \zeta_\tau(\frac{1}{r^2}),$$

$$\left(\Psi''_\tau r^2 + \Psi'_\tau r - \Psi_\tau\right)\Psi'^2_\tau r^2 = \mu_\tau(r^2)r = \nu_\tau(\frac{1}{r^2})\frac{1}{r},$$

where the functions ξ_τ, ζ_τ, ν_τ, μ_τ are of class C^∞ and $\xi_\tau(0) \neq 0,\ \zeta_\tau(0) \neq 0$.

Proof. Initial value problem (1) can be replaced by another initial value problem. The change of variables $r \to s = r^{-2}$, $u \to g = r^{-1}u$ changes (1) to the following initial value problem

$$g''' = 3\frac{g''g'}{g - 2g's} + s\frac{g''^2}{g - 2g's},\ g(1) = 0, g'(1) = -\frac{1}{2}, g''(1) = \tau \tag{9}$$

We will prove that solutions of (9) for parameter $\tau \in (-T, T)$ satisfy the regular initial conditions at $s = 0$: $g(0) \neq 0, g'(0) < \infty, 0 < g''(0) < \infty$.

Compute now for a solution of (3):

$$x'(t) = (\exp t)(g - 2g's), \tag{10}$$

$$x''(t) - x(t) = 4\exp(-3t)g''(s), \tag{11}$$

remember that $s = \exp(-2t)$.

Let us consider system (4). Since the eigenvalues of the corresponding linear system are equal to -2 and -4, there exist functions P and Q of class C^1 such that

$$q = Q(s) = 1 + Cs + o_1(s),$$

$$p = P(s) = -2Cs + o_2(s),$$

where C is a constant, see [4].
We may write:

$$R(t) = \int^t q = t + \psi_1(s).$$

Let us show that ψ_1 is of class C^1. By differentiation we obtain

$$\frac{d\psi_1(s)}{ds} = -\frac{q-1}{2s} = -\frac{Cs + o_1(s)}{2s} \in C^0.$$

Write now for a solution of (3)

$$x(t) = \exp R(t) = (\exp t)\exp\psi_1(s) = (\exp t)g(s),$$

where we have used that $g = r^{-1}u = \exp(-t)u = \exp(-t)x(t)$, and, therefore, $g(0) = \exp\psi_1(0) \neq 0$.

For a solution of (3) which can be extended into infinity it holds

$$x'(t) = x(t)q(t) = (\exp t)(\exp \psi_1(s))(1 + Cs + o_1(s)) =$$

$$= (\exp t)g(s)(1 + Cs + o_1(s)), s = \exp(-2t).$$

On the other hand, from (10) we obtain

$$g(s)(1 + Cs + o_1(s)) = g(s) - 2sg'(s).$$

Thus, for any solution of (3) where the corresponding orbit of (4) converges to the singular point $q = 1, p = 0$ it holds $g'(0) < \infty$.

Let us show that for these solutions it holds $0 < g''(0) < \infty$.
We compute

$$x''(t) - x(t) = x(t)(p + q^2 - 1) = (\exp t)(\exp \psi_1(s))(P(s) + Q^2(s) - 1) =$$

$$= (\exp t)(\exp \psi_1(s))(-2Cs + o_2(s) + (1 + Cs + o_1(s))^2 - 1).$$

Thus,

$$x''(t) - x(t) = (\exp t)\psi_2(s),$$

where $\psi_2 \in C^1$, $\psi_2(s) = o_3(s)$. Rewrite the differential equation in (3) in the following form

$$(x''' - x')x' = -3(x'' - x)x - 2(x'' - x)^2.$$

We obtain either $x''(t) - x(t) \equiv 0$ or

$$(\log(x''(t) - x(t)))' = -3\frac{1}{q} - 2(x''(t) - x(t))(\exp t \exp \psi_1(s)(1 + Cs + o_1(s)))^{-1} =$$

$$= -3\frac{1}{q} - 2\psi_2(s)(\exp \psi_1(s)(1 + Cs + o_1(s)))^{-1}.$$

So,

$$(\log(x''(t) - x(t)))' = -3(1 - Cs) + o_4(s).$$

Then it follows

$$(\log((\exp t)\psi_2(s)))' = -3(1 - Cs) + o_4(s).$$

Thus,

$$(t + \log \psi_2(s))' = -3(1 - Cs) + o_4(s).$$

By integrating we get

$$(\log \psi_2(s)) = -4t - \frac{3C}{2}s + \psi_3(s),$$

where $\psi_3 \in C^1$.
Thus,

$$\psi_2(s) = (\exp(-4t)) \exp(-\frac{3C}{2}s + \psi_3(s)) = s^2\psi_4(s),$$

since $s = \exp(-2t)$. So, $\psi_4 \in C^1$ and $\psi_4(0) = const \neq 0$. Taking into account (11), we obtain $g''(0) = \frac{1}{4}\psi_4(0)$ and therefore $0 < g''(0) < \infty$.
So, we conclude that the solutions of (9) for $\tau \in (-T, T)$ are of class C^∞ in zero.

Let $u(r) = \Psi_\tau(r)$ and $x(t) = \Theta_\tau(t)$ be solutions of (1) and (3) respectively. Then we get that $\Psi_\tau(r) = \Theta_\tau(\log r)$ and

$$\Psi'_\tau(r) = \Theta'(\log r)\frac{1}{r} = g(\frac{1}{r^2}) - 2g'(\frac{1}{r^2})\frac{1}{r^2} = \zeta_\tau(\frac{1}{r^2}),$$

where ζ_τ is of class C^∞. From (10) and the condition $g(0) \neq 0$ it follows that $\zeta_\tau(0) \neq 0$.
Using (10) and (11) we may compute

$$\left(\Psi''_\tau r^2 + \Psi'_\tau r - \Psi_\tau\right)\Psi'^2_\tau r^2 = \left(\Theta''_\tau(t) - \Theta_\tau(t)\right)\Theta'^2_\tau(t) =$$

$$4\exp(-t)g''(s)(g - 2g'(s)s)^2 = 4\frac{1}{r}\left(g(\frac{1}{r^2}) - 2g'(\frac{1}{r^2})\frac{1}{r^2}\right)^2 g''(\frac{1}{r}) = \nu_\tau(\frac{1}{r^2})\frac{1}{r},$$

where ν_τ is of class C^∞.
Since a solution $x(t) = \Theta_\tau(t)$ of (3) equals $-\Theta_{-\tau}(-t)$ if $t \leq 0$, in the same way, we obtain

$$\Psi'_\tau(r) = \frac{1}{r^2}\xi_\tau(r^2),$$

where $\xi_\tau \in C^\infty$, $\xi_\tau(0) \neq 0$ and

$$\left(\Psi''_\tau r^2 + \Psi'_\tau r - \Psi_\tau\right)\Psi'^2_\tau r^2 = \mu_\tau(r^2)r,$$

where μ_τ is of class C^∞. $\square$

4 Smooth conservative systems on S^2

In this chapter we propose new examples of complete integrable conservative systems on S^2 in terms of solutions of (1).

Theorem 4.1 *The one-parameter family of conservative systems with energy (2), where $\Psi \in \{\Psi_\tau\}_{\tau\in(0,T)}$ and Ψ_τ is a solution of (1), is a one-parameter family of smooth systems on S^2 possessing an integral cubic in momenta which is nontrivial, i.e. there is no quadratic or linear integral.*
These systems cannot be obtained one from another by a change of variables.
This family of integrable conservative systems on S^2 does not include the case of Goryachev-Chaplygin.

Proof. Prove now that the kinetic energy of (2) for any $\tau \in (0,T)$ is a C^∞ metric on S^2. Using theorem 3.1, we may write the kinetic energy as

$$\frac{r^2 d\varphi^2 + dr^2}{\xi_\tau(r^2)} = \frac{\tilde r^2 d\tilde\varphi^2 + d\tilde r^2}{\zeta_\tau(\tilde r^2)},$$

where $\tilde r = \frac{1}{r}$, $\tilde\varphi = -\varphi$. Since functions ξ_τ and ζ_τ are of class C^∞ and, moreover, $\xi_\tau(0) \neq 0$, $\zeta_\tau(0) \neq 0$, it follows that the kinetic energy is a smooth metric on S^2.
The potential of (2) also can be written in polar coordinates:

$$V_\tau = \mu_\tau(r^2)r\cos\varphi = \nu_\tau(\tilde r^2)\tilde r\cos\tilde\varphi.$$

Thus, the potential is also a smooth function in $r\cos\varphi$, $r\sin\varphi$, $0 \le r < \infty$ and in $\tilde r\cos\tilde\varphi$, $\tilde r\sin\tilde\varphi$, $0 \le \tilde r < \infty$ and therefore, a smooth function on S^2. So, any system with energy (2) is a C^∞ conservative system on S^2.

We prove now the integrability of these systems.
As mentioned above, if Ψ_τ satisfies (1) then the function $\Theta_\tau = \Psi_\tau \circ \exp$ satisfies (3). In the coordinate system $y = \log r$, $\varphi = \varphi$ energy (2) has the following form:

$$H_\tau = {\Theta_\tau}'^2(y)(p_\varphi^2 + p_y^2) - {\Theta_\tau}'^2(y)({\Theta_\tau}''(y) - \Theta_\tau(y))\cos\varphi, \tag{12}$$

where $\Theta_\tau \in \{\Theta_\tau\}_{\tau\in(0,T)}$.
We will prove that the following polynomial cubic in momenta

$$F_\tau = p_\varphi^3 + \frac{3}{2}\left(\Theta_\tau(y)\cos\varphi p_\varphi - \Theta_\tau'(y)\sin\varphi p_y\right) = p_\varphi^3 + E_\tau(\varphi, y, p_\varphi, p_y)$$

is an integral of the system with the Hamiltonian (12), i.e. $\{F_\tau, H_\tau\} \equiv 0$.
Denote by $\hat H_\tau = {\Theta_\tau}'^2(y)(p_\varphi^2 + p_y^2)$ the Hamiltonian of the geodesic flow of metric

$$\hat{ds_\tau}^2 = \frac{1}{{\Theta_\tau}'^2(y)}\left(d\varphi^2 + dy^2\right).$$

Write now energy as $H_\tau = \hat H_\tau + V_\tau$ where the potential V_τ has the following form

$$V_\tau = -{\Theta_\tau}'^2(y)({\Theta_\tau}''(y) - \Theta_\tau(y))\cos\varphi. \tag{13}$$

Note that $\{\hat H, p_\varphi^3\} \equiv 0$. In order to prove that $\{F_\tau, H_\tau\} \equiv 0$ we must prove that $\{V_\tau, p_\varphi^3\} + \{\hat H_\tau, E_\tau(\varphi, y, p_\varphi, p_y)\} \equiv 0$ and $\{V_\tau, E_\tau(\varphi, y, p_\varphi, p_y)\} \equiv 0$.
Compute

$$\{V_\tau, p_\varphi^3\} + \{\hat H_\tau, E_\tau(\varphi, y, p_\varphi, p_y)\} = -3{\Theta_\tau}'^2({\Theta_\tau}'' - \Theta_\tau)\sin\varphi p_\varphi^2 - 3{\Theta_\tau}'^2(\Theta_\tau \sin\varphi p_\varphi +$$

$${\Theta_\tau}'\cos\varphi p_y)p_\varphi + 3{\Theta_\tau}'^2({\Theta_\tau}'\cos\varphi p_\varphi - {\Theta_\tau}''\sin\varphi p_y)p_y + 3{\Theta_\tau}'^2{\Theta_\tau}''\sin\varphi(p_\varphi^2 + p_y^2) \equiv 0.$$

By a computation we obtain $\{V_\tau, E_\tau(\varphi, y, p_\varphi, p_y)\} =$

$$-\frac{3}{2}\cos\varphi\sin\varphi\Theta_\tau'^2\left(\Theta_\tau'\Theta_\tau''' - \Theta_\tau^2 - \Theta_\tau''\Theta_\tau - \Theta_\tau'^2 + 2\Theta_\tau''^2\right) \equiv 0.$$

Let us assume that a system of this family has an integral which is independent of the energy and which is a polynomial of second degree with respect to momenta (clearly, this assumption includes the case of linear integrals). So, there is an integral $\tilde F_\tau$ of (12) which is quadratic in momenta. Thus, $\tilde F_\tau = A_\tau(p_\varphi, p_y, \varphi, y) + B_\tau(\varphi, y)$ where $A_\tau(p_\varphi, p_y, \varphi, y)$ is a polynomial of second degree with respect to momenta. We may write $\{\tilde F_\tau, H_\tau\} = \{A_\tau(p_\varphi, p_y, \varphi, y) + B_\tau(\varphi, y), \hat H_\tau + V_\tau\} \equiv 0$ and, therefore, $\{A_\tau(p_\varphi, p_y, \varphi, y), \hat H_\tau\} \equiv 0$.
Thus, the geodesic flow of $\hat{ds_\tau}^2$ has an integral which is a polynomial of second

degree with respect to momenta and, therefore, in some Liouville coordinate system u, v it holds

$$\hat{H}_\tau = \frac{1}{\alpha_1(u) + \alpha_2(v)}(p_u^2 + p_v^2), V_\tau = \frac{\alpha_3(u) + \alpha_4(v)}{\alpha_1(u) + \alpha_2(v)} \tag{14}$$

where α_1, α_2, α_3, α_4 are some functions.
Geodesic flows on S^2 possesing an integral quadratic in momenta have been described in [5]. In particular, it has been shown that Liouville coordinate systems for all metrics on S^2 of nonconstant curvature are unique up to linear combinations. Noting that (13) for any $\tau \in (0, T)$ has not form (14) in any coordinate systems $ay + b$, $a\varphi + d$, where a, b, d are some constants, we prove that no system from our series possesses an integral quadratic in momenta. Moreover, since the solutions of (3) for $\tau \in (0, T)$ cannot be obtained one from another by a linear change of variables, systems with the energy (2) also cannot be obtained one from another by a change of variables.

Prove that our family of smooth conservative systems on S^2 does not include the well-known case of Goryachev-Chaplygin from the dynamics of a rigid body.
In the case of Goryachev-Chaplygin there is also a coordinate system φ, y where energy has the following form

$$H = \gamma(y)(p_\varphi^2 + p_y^2) + \beta(y)\cos\varphi$$

for some functions γ and β, see [2]. Since $\hat{H} = \gamma(y)(p_\varphi^2 + p_y^2)$ is the Hamiltonian of the geodesic flow of a smooth metric on S^2, this coordinate system is unique up to linear combinations.
On the other hand there is an integral $F = p_\varphi^3 + \kappa p_\varphi \hat{H} + G(p_\varphi, p_y, \varphi, y)$ where constant $\kappa \neq 0$ and $G(p_\varphi, p_y, \varphi, y)$ is a polynomial linear in momenta (A.V. Bolsinov, private communication), see also [2].
If our series includes the case of Goryachev-Chaplygin then it follows that in the case of Gorychev-Chaplygin there are two independent first integrals. So, our series of integrable conservative systems on S^2 does not include the case of Goryachev-Chaplygin. $\square$

Commentar. Numerically we got that the value of T is equal to ≈0,57735.

Acknowledgement. The author wishes to thank A.V. Bolsinov, A.T. Fomenko and A.M. Stepin for their interest in her work.
The author would like to thank Prof. G. Huisken and Arbeitsbereich "Analysis" of Eberhard-Karls-Universität Tübingen for their hospitality.

References

[1] Babenko I.K., Nekhoroshev N.N. *On complex structures on two dimensional tori admittings metrics with nontrivial quadratic integral,* Math. Notes 58, N5, 1129-1135 (1995)

[2] Bolsinov A.V., Kozlov V.V., Fomenko, A.T. *The Maupertuis principle and geodesic flows on S^2 arising from integrable cases in the dynamics of a rigid body,* Russ. Math. Surv. 50, N3, 473-501 (1995)

[3] Darboux G. *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal.* Paris, Gauthier-Villar, 1896

[4] Hartman, P. *Ordinary differential equations.* New York-London-Sydney, John Wiley and Sons, 1964

[5] Kolokol'tsov, V.N. *Geodesic flows on two-dimensional manifolds with according to the velocities complementary polynomial first integral,* Izv. Akad. Nauk SSSR, Ser. Mat. 46, N5, 994-1010 (1982)

[6] Kozlov V.V. *Integrable and Nonintegrable Hamiltonian systems,* Sov. Sci. Rev. C. Math. Phys. V.8., 1-81 (1988)

Mathematisches Institut, Universität Tübingen
Auf der Morgenstelle 10, 72076 Tübingen, Germany
e-mail: lena@moebius.mathematik.uni-tuebingen.de

Chair of Geometry, Nizhny Novgorod State Pedagogical University
603000 Russia, Nizhny Novgorod, ul. Ulyanova 1